A Generalization of the Mohr-Coulomb Strength Criterion for Soils

Une généralisation du critère de Mohr-Coulomb pour la résistance des sols

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Summary

The paper proposes a generalization of the Mohr-Coulomb strength hypothesis in which it is assumed that the actual slip surface is rough in contrast to the ideal smooth surface considered in the statics of cohesionless media.

Such an assumption enables the influence of all three principal stresses on the strength of the soil to be accounted for and strength characteristics to be established that are invariable with reference to the stress conditions.

Résumé

Dans ce rapport nous proposons la généralisation de l'hypothèse de Mohr-Coulomb pour la résistance des sols. D'après cette généralisation la surface de glissement réelle est supposée rude, se distinguant ainsi de la surface idéale polie dont s'occupe la mécanique des milieux pulvérulents.

Cette hypothèse permet de tenir compte de l'influence des trois contraintes principales sur la résistance des sols et de déterminer les invariantes par rapport aux conditions de contrainte des caractéristiques de résistance.

For soils that are in the state of ultimate stress, the Mohr strength criterion is almost always applied. According to this criterion the strength of cohesionless or cohesive soil at the point being investigated depends only on the ratio of the maximum and minimum principal stresses σ_1 and σ_3.

In this case the intermediate principal stress σ_2 may have any value provided that $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$ (Sokolovsky, 1954). The Mohr strength criterion in the presence of a rectilinear envelope of the limit circles, i.e., the principal equation applied in the statics of cohesionless media (Sokolovsky, 1954) in a state of incomplete (Berezantzev, 1953) limiting equilibrium, can be written as follows

$$\sigma_1 - \sigma_3 = \sigma_1 + \sigma_3 + 2C \cot \varphi \qquad (1)$$

where C – cohesion
φ – angle of internal friction.

The Mohr strength hypothesis is based on the Coulomb criterion and states that the state of stress at a point becomes limiting when the maximum value of the function $F = |\tau_n| - \sigma_n \tan \varphi - C$ is equal to zero, i.e.,

$$F = \{ |\tau_n| - \sigma_n \tan \varphi \}_{max} - C = 0 \qquad (2)$$

Here, τ_n and σ_n are the tangential and normal stresses acting on an area with an exterior normal n. Failure usually occurs in soils due to the breaking of the bonds between particles and the relative sliding of spearate particles or grains. These bonds can be destroyed either as the result of a mechanical action on the soil, i.e. a change in its state of stress, or due to changes in the physical state of the soil – its density, moisture content and cohesion.

A shortcoming of the Mohr strength hypothesis is the fact that it does not take the intermediate principal stress into consideration. In the Mises-Botkin strength theory in which the intermediate principal stress appears on a par with the maximum and the minimum principal stresses, its influence proves to be too significant, which does not conform with the experimental data of many investigators.

A more useful strength theory would be one conforming more substantially to the results of experiments and therefore physically justified. The mechanical soil strength theory discussed below enables the influence of the changes in the state of stress on the strength of the soil to be determined for a practically constant physical state. Since, as mentioned earlier, breaking of the bonds between particles leads to their mutual sliding and, at moderate pressures, only to a small degree of crushing of contact points and destruction of the particles themselves (Tsytovich, 1963), application of the Coulomb strength criterion is most justified. In particular cases the latter leads to the Mohr strength criterion where the intermediate principal stress is not taken into consideration, and to the Mises-Botkin strength criterion where the intermediate principal stress is given equal importance as the maximum and minimum principal stresses (Malyshev, 1963).

As a rule a homogeneous cohesionless medium is treated by the theory of limiting equilibrium, thereby presuming that slip is kinematically possible precisely along that plane for which condition (2) is valid. Actually a cohesionless medium is not homogeneous in the sense that slip in the medium is only possible along the contacts of the soil particles without the particles themselves being re-orientated, revolved and slightly deformed. The concentration of stresses developed at the contacts between the particles naturally leads to a certain crushing and destruction of the particles themselves at pressures that are of interest to us. This phenomenon is comparatively minute and it is considered that on the whole sliding of the particles on each other occurs along the contact points.

Thus on this basis a cohesionless medium should be regarded as a non-homogeneous medium where the slip of certain particles of soil over others can occur mainly along the points of contact between them.

For slip under conditions of plane strain theoretical slip surfaces are assumed to be cylindrical with a rectilinear generatrix parallel to the axis of the principal stresses σ_2 (Fig. 1).

Therefore, the generatrix is a straight line, whose direction coincides with axis σ_2 (Fig. 1). In practice this will not be a true straight line but a broken one since it should be drawn around the particles and along the contacts without intersecting the particles (Fig. 2a). The slip planes deviate from this line at different angles whose value is naturally not known for each individual case. For example, it can be assumed (Malyshev, 1963) that the angle of inclination Δ has an equal probability of occuring the interval from $-\Delta_1$ to $+\Delta_1$ where $\pm\Delta_1$ is the maximum angle of inclination. Thus, all the intermediate values of the angles are included within the limits $-\Delta_1 \leqslant \Delta \leqslant \Delta_1$.

However, such an equal probability can be assumed only at the initial moment before displacement since reorientation of the particles occurs subsequently and this concept will no longer be valid, i.e., the function expressing the distribution of these angles will vary during the process of displacement.

It is natural to assume that the values of the angles of inclination Δ are reduced by the re-orientation of the particles; this corresponds to loosening of the soil during displacement. If the random angle of inclination of the slip plane is denoted as Δ and, as mentioned above $-\Delta_1 \leqslant \Delta \leqslant \Delta_1$, then we obtain the equation of a circular arc (Fig. 2 b) in the case of equal probability of inclination angle. Subsequently, on displacement the number of planes with low values Δ will increase and that with high values Δ will decrease. The Mohr relationship corresponds to the assumption that for all planes having an inclination of $\Delta = 0$ the probability is equal to unity, while for the case $\Delta \neq 0$ the probability is equal to zero.

In the following, for simplicity, we will consider the case of an equally probable inclination of the planes in the direction (Fig. 3) coinciding with axis σ_2.

As is known, the normal stress on any plane with a normal n, σ_n, and the tangential stress τ_n acting on the same plane are equal to

$$\begin{aligned} \sigma_n &= (\sigma_1 - \sigma_3)\, l^2 + (\sigma_2 - \sigma_3)\, m^2 + \sigma_3 \\ \tau_n^2 &= (\sigma_1^2 - \sigma_3^2)\, l^2 + (\sigma_2^2 - \sigma_3^2)\, m^2 + \sigma_3^2 - \sigma_n^2 \end{aligned} \tag{3}$$

Here, l and m are the direction cosines of the angles made by the principal stress directions with the normal n (see Fig. 1).

Substituting equation (3) into equation (2) and determining the maximum by equating the partial derivatives $\partial F/\partial l$ and $\partial F/\partial m$ to zero we obtain the well-known equation from the statics of cohesionless media (Sokolovsky, 1954) for direction cosines l and m, i.e.

$$l^2 = \cos^2\left(\frac{\pi}{4} + \frac{\varphi}{2}\right) = \frac{1}{2}(1 - \sin\varphi); \quad m = 0 \tag{4}$$

However, in our statement of the problem, instead of substituting the values of the stresses acting on a definite plane with a normal n, in equation (2) we will substitute the

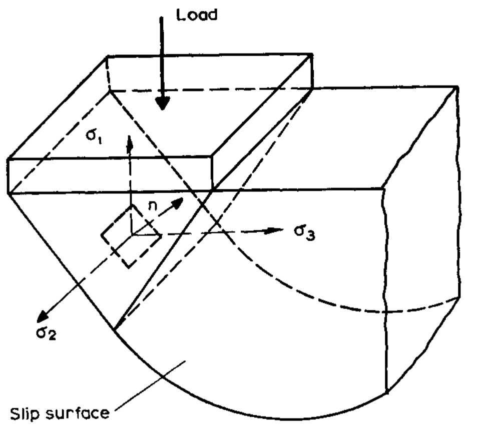

Fig. 1

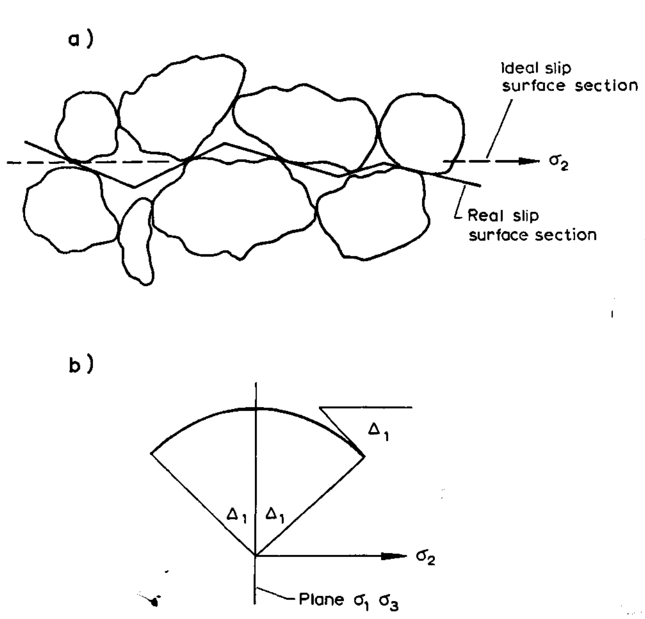

Fig. 2

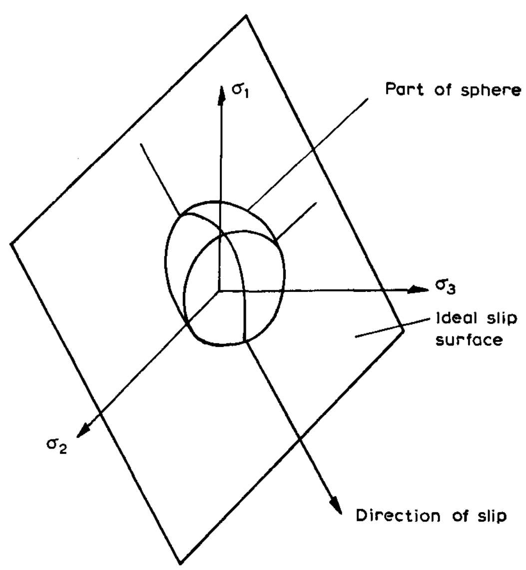

Fig. 3

mean values of the stresses acting within the limits of deviation of the planes from planes of normal n.

This can be clearly seen in Fig. 2 a.

We shall define the mean values of the tangential stresses as

$$\tau_{n,\,av} = \sqrt{\frac{\int_{l_1}^{l_2}\int_{m_1}^{m_2} \tau_n^2 \, dl\, dm}{\int_{l_1}^{l_2}\int_{m_1}^{m_2} dl\, dm}} \tag{5}$$

Here the expression is integrated for the squared value of τ_n because its absolute value is represented in equation (2).

The mean normal stress is defined as

$$\sigma_{n,\,av} = \frac{\int_{l_1}^{l_2}\int_{m_1}^{m_2} \sigma_n \, dl\, dm}{\int_{l_1}^{l_2}\int_{m_1}^{m_2} dl\, dm} \tag{6}$$

Since we have assumed equality of the limiting values $\pm\varDelta_1$ of the inclination in both directions then on the basis of equation (4) we can obtain the following limits of integration shown in equations (5) and (6):

$$l_1 = \cos\left(\frac{\pi}{4} + \frac{\varphi}{2} + \varDelta_1\right) \; ; \quad l_2 = \cos\left(\frac{\pi}{4} + \frac{\varphi}{2} - \varDelta_1\right) \tag{7}$$

$$m_1 = \cos\left(\frac{\pi}{2} + \varDelta_1\right) = -\sin\varDelta_1 \; ; \quad m_2 = \cos\left(\frac{\pi}{2} - \varDelta_1\right) = \sin\varDelta_1$$

Carrying out the integration within these limits we obtain

$$\int_{l_1}^{l_2}\int_{m_1}^{m_2} dl\, dm = 4 \sin\left(\frac{\pi}{4} + \frac{\varphi}{2}\right) \cdot \sin^2\varDelta_1 \tag{8}$$

$$\sigma_{n,\,av} = \frac{1}{6}(\sigma_1 - \sigma_3)\,[2\cos^2\varDelta_1\,(1 - 2\sin\varphi) + 1 + \sin\varphi] + \tag{9}$$

$$+ \frac{1}{3}(\sigma_2 - \sigma_3)\sin^2\varDelta_1 + \sigma_3 = \frac{1}{3}\{\alpha_1\sigma_1 + \alpha_3\sigma_3 + [3 - (\alpha_1 + \alpha_2)]\,\sigma_3\}$$

$$\tau_{n,\,av} = \frac{1}{45}\{3(5\alpha_1 - 3\alpha_2)\,\sigma_1^2 + 3(5\alpha_3 - 3\alpha_4)\,\sigma_2^2 - \tag{10}$$

$$-(9\alpha_2 + 9\alpha_4 + 5\alpha_1\alpha_3)\sigma_3^2 - 5\alpha_1\alpha_3\sigma_1\sigma_2 +$$

$$+ (18\alpha_4 + 5\alpha_1\alpha_3 - 15\alpha_3)\sigma_2\sigma_3 + (18\alpha_2 + 5\alpha_1\alpha_3 - 15\alpha_1)\sigma_1\sigma_3\}$$

$$\alpha_1 = (1 - 2\sin\varphi)\cos^2\varDelta_1 + \frac{1}{2}(1 + \sin\varphi) \tag{11}$$

$$\alpha_2 = (4\sin^2\varphi - 2\sin\varphi - 1)\cos^4\varDelta_1 +$$

$$+ (2 - \sin\varphi - 3\sin^2\varphi)\cos^2\varDelta_1 + \frac{1}{4}(1 + \sin\varphi)^2$$

$$\alpha_3 = \sin^2\varDelta_1 \; ; \quad \alpha_4 = \sin^4\varDelta_1 = \alpha_3^2$$

Since we wish to treat the limiting condition in the case under consideration and $\tau_{n,\,av}$ and $\sigma_{n,\,av}$ refer to the slip planes then equation (2) can be rewritten as

$$\tau_{n,\,av} - \sigma_{n,\,av}\tan\varphi - C \tag{12}$$

$$\tau_{n,\,av}^2 - \sigma_{n,\,av}^2\tan^2\varphi = 2C\,\sigma_{n,\,av}\tan\varphi + C^2 \tag{13}$$

By substituting equations (9) and (10) into equation (13) and transforming, we obtain the following strength criterion

$$a_{11}\,\sigma_1^2 + a_{22}\,\sigma_2^2 + a_{33}\,\sigma_3^2 + a_{12}\,\sigma_1\,\sigma_2 + a_{23}\,\sigma_2\,\sigma_3 +$$

$$+ a_{31}\,\sigma_3\,\sigma_1 = C\,(a_1\,\sigma_1 + a_2\,\sigma_2 + a_3\,\sigma_3) + C^2 \tag{14}$$

In equation (14) the coefficients are expressed as follows:

$$a_{11} = \frac{1}{45}(15\alpha_1 - 9\alpha_2 - 5\alpha_1^2\tan^2\varphi) \tag{15}$$

$$a_{22} = \frac{1}{45}(15\alpha_3 - 9\alpha_4 - 5\alpha_3^2\tan^2\varphi)$$

$$a_{33} = -\frac{1}{45}[9\alpha_2 + 9\alpha_4 + 5\alpha_1\alpha_3 + 5(3 - \alpha_1 - \alpha_3)^2\tan^2\varphi]$$

$$a_{12} = -\frac{1}{9}\alpha_1\alpha_3(1 + 2\tan^2\varphi)$$

$$a_{23} = \frac{1}{45}[18\alpha_4 + 5\alpha_1\alpha_3 - 15\alpha_3 - 10\alpha_3(3 - \alpha_1 - \alpha_3)\tan^2\varphi]$$

$$a_{31} = \frac{1}{45}[18\alpha_2 + 5\alpha_1\alpha_3 - 15\alpha_1 - 10\alpha_1(3 - \alpha_1 - \alpha_3)\tan^2\varphi]$$

$$a_1 = \frac{2}{3}\alpha_1\tan\varphi$$

$$a_2 = \frac{2}{3}\alpha_3\tan\varphi$$

$$a_3 = \frac{2}{3}[3 - (\alpha_1 + \alpha_3)]\tan\varphi = 2\tan\varphi - a_1 - a_2$$

The strength criterion (14) has quite a complex form. In the particular case when $\varDelta_1 = 0$, it simplifies to the well-known Mohr strength criterion (1).

It follows from criterion (14) that it takes account of three principal stresses. Thus, the generalization of the Mohr strength criterion performed above takes into account the influence of the intermediate principal stress on the soil strength. Some investigators (Malyshev, 1954, 1963; Kirkpatrick, 1957; Habib, 1953; Cornforth, 1964; Shibata, 1965) have repeatedly pointed out that the intermediate principal stress influences the soil strength. All the experiments show that this influence is less than that obtained according to the Mises-Botkin strength theory (Botkin, 1940).

The results of the experiment with shot (Malyshev, 1963) are given in Fig. 4. If the Mohr strength hypothesis were valid they would lie at the ends of the ellipse intercepts 5. Actually they occupy an intermediate position nearer to the Mohr than to the Mises-Botkin strength criterion.

An analysis of the results of experiments concerning the proposed strength hypothesis does not depend on the soil porosity, or the parameter Δ_1, which is a function of the porosity.

The internal friction angle that we usually determine is a certain function of angles φ and Δ_1, and coincides with the true angle of internal friction only in the loosest condition of the soil, when $\Delta_1 = 0$.

The larger the angle Δ_1, the larger the apparent angle of internal friction. To determine these parameters it is necessary to conduct two tests with one combination of the principal stresses and one test with another combination of the principal stresses while maintaining the same physical condition of the soil. For cohesionless soils it is necessary to conduct at least two tests at different states of stress. In a triaxial test ($\sigma_2 = \sigma_3$) on cohesionless soil criterion (14) shall be

$$a_{11} + \left(\frac{\sigma_3}{\sigma_1}\right)_t (a_{31} + a_{12}) + \left(\frac{\sigma_3}{\sigma_1}\right)_t^2 (a_{22} + a_{33} + a_{23}) = 0 \quad (16)$$

while for plane strain, when

$$\sigma_2 = \nu\,(\sigma_1 + \sigma_3) \quad (17)$$

then

$$a_{11} + a_{12}\,\nu + a_{22}\,\nu^2 + \left(\frac{\sigma_3}{\sigma_1}\right)_p (2\,a_{22}\,\nu^2 + a_{12}\,\nu + a_{23}\,\nu + a_{31}) + \quad (18)$$

$$+ \left(\frac{\sigma_3}{\sigma_1}\right)_p^2 (a_{22}\,\nu^2 + a_{23}\,\nu + a_{33}) = 0$$

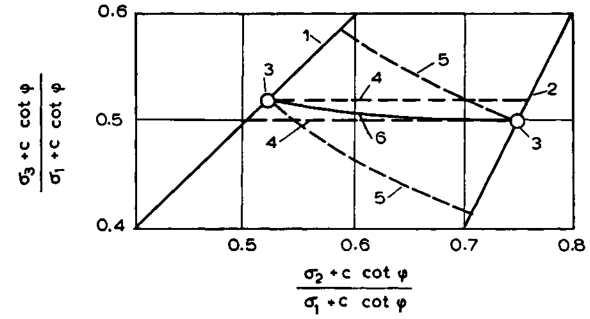

Fig. 4. *Test results for shot.*
1 – Triaxial compression.
2 – Torsion.
3 – Test points – average result of all tests.
4 – Relationship $\dfrac{\sigma_3 + C\cot\varphi}{\sigma_1 + C\cot\varphi} = f\left(\dfrac{\sigma_2 + C\cot\varphi}{\sigma_1 + C\cot\varphi}\right)$ *according to Mohr (ordinary treatment).*
5 – The same relationship according to Mises-Botkin.
6 – The same relationship according to proposed hypothesis.
Résultats d'essais.
1 – Compression triaxiale.
2 – Torsion.
3 – Résultat moyen pour tous les essais.
4 – Relation $\dfrac{\sigma_3 + C\cot\varphi}{\sigma_1 + C\cot\varphi} = f\left(\dfrac{\sigma_2 + C\cot\varphi}{\sigma_1 + C\cot\varphi}\right)$ selon Mohr (méthode classique).
5 – Même relation selon Mises-Botkin.
6 – Même relation selon l'hypothèse proposée.

Since the ratios $(\sigma_3/\sigma_1)_t$ and $(\sigma_3/\sigma_1)_p$, corresponding to soil failure are known from experiments, the values of φ and Δ_1 can be determined by trial and error.

In connection with the complex form of the strength criterion (14) it proves possible to replace it, with an accuracy sufficient for all practical purposes, by a linear equation of the following form

$$A_1\,\sigma_1 + A_2\,\sigma_2 + A_3\,\sigma_3 = C \quad (19)$$

where coefficients A_1, A_2 and A_3 depend upon φ and Δ. Such a simplification is equivalent to the linear approximation of equation (5) with the subsequent use of condition (12).

Conclusions

1. Experimental data show that neither the Mohr strength hypothesis nor the Mises-Botkin hypothesis provide strength characteristics which are unique for all conditions of stress.
2. By introducing the proporsal that the real slip planes do not coincide with the ideal ones, it is possible to take into account the influence of the intermediate principal stress and to obtain strength characteristics which are invariable in reference to the state of stress.

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